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Additivity of Jordan elementary maps on nest algebras

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Abstract

Let $\mathscr{A}$ be a standard subalgebra of a nest algebra on a Hilbert space of dimension greater than one and $\mathscr{B}$ an arbitrary algebra. A Jordan elementary map of $\mathscr{A} \times \mathscr{B}$ is a pair (M, M^*) where $M : \mathscr{A} \to \mathscr{B}$ and $M^* : \mathscr{B} \to \mathscr{A}$ are two maps satisfying

$$\begin{cases} M(AM^*(B)A) = M(A)BM(A), \\ M^*(BM(A)B) = M^*(B)AM^*(B) \end{cases}$$

for $A \in \mathscr{A}$, $B \in \mathscr{B}$. In this note, it is proved that for a special class of surjective Jordan elementary maps of $\mathscr{A} \times \mathscr{B}$, every member in it is automatically additive. Also, we construct a counterexample which shows that this result is not necessarily true for all surjective Jordan elementary maps.

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1. Introduction

Let $\mathscr{A}$, $\mathscr{B}$ be two algebras (or rings) and let $M : \mathscr{A} \to \mathscr{B}$, $M^* : \mathscr{B} \to \mathscr{A}$ be two maps. Call the ordered pair (M, M^*) an *elementary map* of $\mathscr{A} \times \mathscr{B}$ (of length one) if

$$\begin{cases} M(AM^*(B)C) = M(A)BM(C), \\ M^*(BM(A)D) = M^*(B)AM^*(D) \end{cases}$$

for $A, C \in \mathscr{A}$, $B, D \in \mathscr{B}$, and a *Jordan elementary map* of $\mathscr{A} \times \mathscr{B}$ (of length one) if

$$\begin{cases} M(AM^*(B)A) = M(A)BM(A), \\ M^*(BM(A)B) = M^*(B)AM^*(B) \end{cases}$$

for $A \in \mathscr{A}$, $B \in \mathscr{B}$. These concepts come from the recent papers [2,3,14], in which the authors introduced and studied linear elementary maps and linear Jordan elementary maps of algebras. Note that we say that the pair (M, M^*) has some property if both M and M^* have the same property; for example, that (M, M^*) is additive means that M and M^* are both additive. If $\phi : \mathscr{A} \to \mathscr{B}$ is a bijective multiplicative map, then clearly the pair (ϕ, ϕ^{-1}) is an elementary map of $\mathscr{A} \times \mathscr{B}$. For $T, S \in \mathscr{A}$, let $M_{T,S}(A) = TAS$ for $A \in \mathscr{A}$; such a map is usually said to be an elementary operator of length one of $\mathscr{A}$. Obviously, $(M_{T,S}, M_{S,T})$ is an elementary map of $\mathscr{A} \times \mathscr{A}$. Another example of an elementary map of $\mathscr{A} \times \mathscr{A}$ is a double centralizer on $\mathscr{A}$ if $\mathscr{A}$ is faithful [2]. Further, call a map $\phi : \mathscr{A} \to \mathscr{B}$ a *Jordan triple map* if

$$\phi(ABA) = \phi(A)\phi(B)\phi(A)$$

for $A, B \in \mathscr{A}$ [1]. It is easily seen that every bijective Jordan triple map $\phi : \mathscr{A} \to \mathscr{B}$ gives rise to a Jordan elementary map (ϕ, ϕ^{-1}) of $\mathscr{A} \times \mathscr{B}$.

Characterizing the interrelation between the multiplicative and the additive structures of a ring is an interesting topic. The first quite surprising result on how the multiplicative structure of a ring determines its additive structure is due to Martindale who established a condition on a ring such that every bijective multiplicative map on this ring is additive [11, Theorem]. Recently, we extended Martindale's result to elementary maps of rings [8, Theorem 2.1]. In particular, if $\mathscr{R}$ is a prime ring containing a non-trivial idempotent and $\mathscr{R}'$ an arbitrary ring, then every surjective elementary map of $\mathscr{R} \times \mathscr{R}'$ is additive [8, Corollary 2.1], where we remark that $\mathscr{R}$ can be taken to be a standard operator algebra on a Banach space of dimension greater than one [8, Corollary 2.2] (see also [14] if $\mathscr{R}'$ is also a standard operator algebra, but with a different proof). In addition, in [12] Molnár described the form of a bijective Jordan triple map between standard operator algebras on Banach spaces of dimensions greater than two; it turns out that such a map is additive. We in [7] generalized the "additive part" of Molnár's result quite significantly, proving that the following.

Proposition 1.1. *If $\mathscr{R}$ is a 2-torsion free prime ring containing a non-trivial idempotent and $\mathscr{R}'$ an arbitrary ring, then every surjective Jordan elementary map of $\mathscr{R} \times \mathscr{R}'$ is additive.*

Nest algebras are the natural analogues of upper triangular matrix algebras in infinite dimensional spaces and an important object of the class of non-selfadjoint operator algebras. Generally, a standard subalgebra of a nest algebra is not a prime ring and might have no idempotents. It is worth to note that the results mentioned in the preceding paragraph heavily depend on idempotents. However, the second author in [9,10] successfully proved that the additivity of bijective r-Jordan maps and bijective multiplicative maps between standard subalgebras of nest algebras on Hilbert spaces of dimensions greater than one, respectively. Here, r is a non-zero rational number and a r-Jordan map is a map ϕ from an algebra $\mathscr{A}$ into another satisfying

$$\phi(r(AB + BA)) = r(\phi(A)\phi(B) + \phi(B)\phi(A))$$

for $A, B \in \mathscr{A}$. Especially, for several historical results concerning the additivity of $\frac{1}{2}$-Jordan maps on operator algebras, see [5,6,13].

The above-mentioned results lead naturally to the following.

Question. Let $\mathscr{A}$ and $\mathscr{B}$ be standard subalgebras of nest algebras on Hilbert spaces of dimensions greater than one. Is every surjective Jordan elementary map of $\mathscr{A} \times \mathscr{B}$ additive?

In this note we will shows that the answer to this question is generally "no", but "yes" for a special class of surjective Jordan elementary maps of $\mathscr{A} \times \mathscr{B}$, in which every member (M, M^*) satisfies

$$\begin{cases} M\left(\frac{1}{2}(AM^*(B)C + CM^*(B)A)\right) \\ \quad = \frac{1}{2}(M(A)BM(C) + M(C)BM(A)), \\ M^*\left(\frac{1}{2}(BM(A)D + DM(A)B)\right) \\ \quad = \frac{1}{2}(M^*(B)AM^*(D) + M^*(D)AM^*(B)) \end{cases} \tag{1.1}$$

for $A, C \in \mathscr{A}$, $B, D \in \mathscr{B}$. An obvious example of such a Jordan elementary map is the pair (ϕ, ϕ^{-1}) where $\phi : \mathscr{A} \to \mathscr{B}$ is a bijective map such that

$$\phi\left(\frac{1}{2}(ABC + CBA)\right) = \frac{1}{2}\left(\phi(A)\phi(B)\phi(C) + \phi(C)\phi(B)\phi(A)\right)$$

for $A, B, C \in \mathscr{A}$. If $A = C$ then it is a Jordan triple map.

Now let us introduce the notation and the concepts, each of which has a natural analogue on a Banach space. Let H be a Hilbert space and denote by $\mathscr{B}(H)$ the algebra of all bounded linear operators on H. For $x, y \in H$, the operator $x \otimes y$ is defined by $z \mapsto (z, y)x (z \in H)$, where $(\cdot, \cdot)$ denotes the inner product. A *nest* on H is a chain of orthogonal projections in $\mathscr{B}(H)$ containing 0 and the identity operator I

which is closed in the strong operator topology. Given a nest $\mathcal{N}$ on H, the associated *nest algebra* Alg$\mathcal{N}$ is defined by

$$\mathrm{Alg}\mathcal{N} = \{T \in B(H) : TP = PTP, \forall P \in \mathcal{N}\}.$$

If $\mathcal{N}$ is trivial, that is $\mathcal{N} = \{0, I\}$, then $\mathrm{Alg}\mathcal{N} = \mathscr{B}(H)$. Also, $\mathcal{N} \subseteq \mathrm{Alg}\mathcal{N}$. Recalling that a standard operator algebra on H is a subalgebra $\mathscr{A} \subseteq \mathscr{B}(H)$ containing all finite rank operators in $\mathscr{B}(H)$, we similarly call a subalgebra $\mathscr{A} \subseteq \mathrm{Alg}\mathcal{N}$ an *standard subalgebra* of Alg$\mathcal{N}$ if it contains all finite rank operators in Alg$\mathcal{N}$. For a general discussion of nest algebras we refer to [4], in which the following basic properties concerning nest algebras can be found.

Lemma 1.1. *Let $\mathcal{N}$ be a nest on a Hilbert space H. Then*

(1) *the rank one operator $x \otimes y \in \mathrm{Alg}\mathcal{N}$ if and only if there exists $P \in \mathcal{N}$ such that $x \in PH$, $y \in (I - P_-)H$, where $P_- = \sup\{Q \in \mathcal{N} : Q < P\}$;*
(2) *the span of the rank one operators in* Alg$\mathcal{N}$ *is dense in* Alg$\mathcal{N}$ *in the strong operator topology.*

2. Main results

We first construct a counterexample which shows that the answer to the "Question" is not necessarily affirmative for all surjective Jordan elementary maps.

Example 2.1. Let $\mathscr{A}$ be a non-trivial nest algebra acting on a real or complex Hilbert space of dimension two. Then $\mathscr{A}$ can be considered to be the algebra of all 2×2 upper triangular matrices over the real field or the complex field. Define the map $\phi : \mathscr{A} \to \mathscr{A}$ by

$$\phi\left(\begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 2z \\ 0 & 0 \end{bmatrix} \text{ and } \phi\left(\begin{bmatrix} x & z \\ 0 & y \end{bmatrix}\right) = \begin{bmatrix} x & z \\ 0 & y \end{bmatrix} \text{ if } |x| + |y| \neq 0.$$

We have

(1) ϕ is bijective;
(2) ϕ is not additive, and hence neither is ϕ^{-1};
(3) the pair (ϕ, ϕ^{-1}) is a Jordan elementary map of $\mathscr{A} \times \mathscr{A}$.

Proof. We only prove (3) since (1) and (2) are obvious. Because of the reason remarked in Section 1, it suffices to verify that ϕ is a Jordan triple map, that is, $\phi(ABA) = \phi(A)\phi(B)\phi(A)$ holds for all $A, B \in \mathscr{A}$.

Let $A = \begin{bmatrix} x & z \\ 0 & y \end{bmatrix}$ and $B = \begin{bmatrix} u & w \\ 0 & v \end{bmatrix}$. A direct computation gives

$$ABA = \begin{bmatrix} x^2u & xzu + xyw + yzv \\ 0 & y^2v \end{bmatrix}. \tag{2.1}$$

For our purpose, we distinguish three cases.

Case 1. $|x| + |y| = 0$. Observing that $ABA = 0$ and $\phi(A) = \begin{bmatrix} 0 & 2z \\ 0 & 0 \end{bmatrix}$, it is easily verified that $\phi(ABA) = 0 = \phi(A)\phi(B)\phi(A)$.

Case 2. $|x| + |y| \neq 0$ and $|u| + |v| = 0$. Then $ABA = \begin{bmatrix} 0 & xyw \\ 0 & 0 \end{bmatrix}$ by (2.1), and hence $\phi(ABA) = \begin{bmatrix} 0 & 2xyw \\ 0 & 0 \end{bmatrix}$. Noting that $\phi(A) = A$ and $\phi(B) = \begin{bmatrix} 0 & 2w \\ 0 & 0 \end{bmatrix}$, it follows that $\phi(A)\phi(B)\phi(A) = \begin{bmatrix} 0 & 2xyw \\ 0 & 0 \end{bmatrix}$. So $\phi(ABA) = \phi(A)\phi(B)$ $\phi(A)$.

Case 3. $|x| + |y| \neq 0$ and $|u| + |v| \neq 0$. Then $\phi(A) = A$ and $\phi(B) = B$ and so $\phi(A)\phi(B)\phi(A) = ABA$. On the other hand, if $|x^2u| + |y^2v| \neq 0$, then $\phi(ABA) = ABA$ by (2.1). If $|x^2u| + |y^2v| = 0$, then either $x = v = 0$ or $u = y = 0$; moreover, we always have $\phi(ABA) = 0 = ABA$ by (2.1) again. Therefore $\phi(ABA) = \phi(A)\phi(B)\phi(A)$. The proof is complete. □

Now we will prove that the answer to the "Question" is always affirmative for every surjective Jordan elementary map satisfying (1.1). Our result reads as follows.

Theorem 2.1. *Let H be a real or complex Hilbert space with* $\dim H > 1$, *$\mathscr{N}$ be a nest on H, $\mathscr{A}$ be a standard subalgebra of* $\operatorname{Alg}\mathscr{N}$ *and $\mathscr{B}$ be an arbitrary algebra. Suppose that $M : \mathscr{A} \to \mathscr{B}$ and $M^* : \mathscr{B} \to \mathscr{A}$ are surjective maps such that* (1.1) *holds for all $A, C \in \mathscr{A}$, $B, D \in \mathscr{B}$. Then both M and M^* are additive.*

Throughout what follows, the notation of the theorem above will be kept and we write $\mathscr{F}$ for the span of the rank one operators in $\operatorname{Alg}\mathscr{N}$, which is a subalgebra of $\mathscr{A}$. For clarity of exposition, we shall organize the proof of Theorem 2.1 in a series of lemmas, in which $\mathscr{N}$ is assumed to be non-trivial. Choose an element $P_1 \in \mathscr{N}$ such that $0 < P_1 < I$ and denote by $P_2 = I - P_1$. Put $\mathscr{F}_{11} = P_1\mathscr{F}P_1$, $\mathscr{F}_{12} = P_1\mathscr{F}P_2$ and $\mathscr{F}_{22} = P_2\mathscr{F}P_2$. Since $P_2AP_1 = 0$ for all $A \in \mathscr{A}$ and $\mathscr{F}$ is an ideal of $\operatorname{Alg}\mathscr{N}$, we have $\mathscr{F} = \mathscr{F}_{11} \oplus \mathscr{F}_{12} \oplus \mathscr{F}_{22}$.

Let us begin with a trivial lemma and omit the proof.

Lemma 2.1. $M(0) = 0$ *and* $M^*(0) = 0$.

Lemma 2.2. *Both M and M^* are bijective.*

Proof. It suffices to prove that M and M^* are injective. First establish the injectivity of M. Suppose that $A_1, A_2 \in \mathscr{A}$ such that $M(A_1) = M(A_2)$. Let $X, Y \in \mathscr{F}$ be arbitrary. Since M^* is surjective, we can choose $B, D \in \mathscr{B}$ such that $M^*(B) = X$, $M^*(D) = Y$. Applying (1.1) we have

$$
\begin{aligned}
\frac{1}{2}(XA_1Y + YA_1X) &= M^*\left(\frac{1}{2}(BM(A_1)D + DM(A_1)B)\right) \\
&= M^*\left(\frac{1}{2}(BM(A_2)D + DM(A_2)B)\right) \\
&= \frac{1}{2}(XA_2Y + YA_2X).
\end{aligned}
$$

Consequently, $A_1 = A_2$ by Lemma 1.1 (2).

To prove the injectivity of M^*, let $B_1, B_2 \in \mathscr{B}$ such that $M^*(B_1) = M^*(B_2)$ and let X, Y be as above. Since M^*M is also surjective, there exist $X', Y' \in \mathscr{A}$ satisfying $M^*M(X') = X$, $M^*M(Y') = Y$. We then have by (1.1) again

$$
\begin{aligned}
&\frac{1}{2}\left(XM^{-1}(B_1)Y + YM^{-1}(B_1)X\right) \\
&\quad = M^*\left(\frac{1}{2}\left(M(X')MM^{-1}(B_1)M(Y') + M(Y')MM^{-1}(B_1)M(X')\right)\right) \\
&\quad = M^*M\left(\frac{1}{2}\left(X'M^*(B_1)Y' + Y'M^*(B_1)X'\right)\right) \\
&\quad = M^*M\left(\frac{1}{2}\left(X'M^*(B_2)Y' + Y'M^*(B_2)X'\right)\right) \\
&\quad = \frac{1}{2}\left(XM^{-1}(B_2)Y + YM^{-1}(B_2)X\right).
\end{aligned}
$$

It follows from Lemma 1.1 (2) that $M^{-1}(B_1) = M^{-1}(B_2)$ and so $B_1 = B_2$. This completes the proof. $\square$

Lemma 2.3. *For $A, C \in \mathscr{A}$ and $B, D \in \mathscr{B}$, the pair $(M^{*^{-1}}, M^{-1})$ satisfies*

$$
\begin{cases}
M^{*^{-1}}\left(\frac{1}{2}\left(AM^{-1}(B)C + CM^{-1}(B)A\right)\right) \\
\quad = \frac{1}{2}\left(M^{*^{-1}}(A)BM^{*^{-1}}(C) + M^{*^{-1}}(C)BM^{*^{-1}}(A)\right), \\
M^{-1}\left(\frac{1}{2}\left(BM^{*^{-1}}(A)D + DM^{*^{-1}}(A)B\right)\right) \\
\quad = \frac{1}{2}\left(M^{-1}(B)AM^{-1}(D) + M^{-1}(D)AM^{-1}(B)\right).
\end{cases}
\tag{2.2}
$$

Proof. By (1.1) we have

$$
\begin{aligned}
&M^*\left(\frac{1}{2}\left(M^{*^{-1}}(A)BM^{*^{-1}}(C) + M^{*^{-1}}(C)BM^{*^{-1}}(A)\right)\right) \\
&\quad = M^*\left(\frac{1}{2}\left(M^{*^{-1}}(A)MM^{-1}(B)M^{*^{-1}}(C) + M^{*^{-1}}(C)MM^{-1}(B)M^{*^{-1}}(A)\right)\right) \\
&\quad = \frac{1}{2}\left(AM^{-1}(B)C + CM^{-1}(B)A\right)
\end{aligned}
$$

and so the first equality holds. The second equality follows similarly. $\square$

Lemma 2.4. *If $S, A, B \in \mathscr{A}$ such that $M(S) = M(A) + M(B)$, then for all X, $Y \in \mathscr{A}$*

(1) $M(SXY + YXS) = M(AXY + YXA) + M(BXY + YXB)$;
(2) $M^{*^{-1}}(XSY + YSX) = M^{*^{-1}}(XAY + YAX) + M^{*^{-1}}(XBY + YBX)$.

Proof. Let $X, Y \in \mathscr{A}$. Then by (1.1)

$$\begin{aligned}
M(SXY + YXS) &= M\left(\frac{1}{2}\left(SM^*M^{*^{-1}}(2X)Y + YM^*M^{*^{-1}}(2X)S\right)\right)\\
&= \frac{1}{2}\left(M(S)M^{*^{-1}}(2X)M(Y) + M(Y)M^{*^{-1}}(2X)M(S)\right)\\
&= \frac{1}{2}\left(M(A) + M(B)\right) M^{*^{-1}}(2X)M(Y)\\
&\quad + \frac{1}{2}M(Y)M^{*^{-1}}(2X)(M(A) + M(B))\\
&= \frac{1}{2}M(A)M^{*^{-1}}(2X)M(Y) + \frac{1}{2}M(Y)M^{*^{-1}}(2X)M(A)\\
&\quad + \frac{1}{2}M(B)M^{*^{-1}}(2X)M(Y) + \frac{1}{2}M(Y)M^{*^{-1}}(2X)M(B)\\
&= M(AXY + YXA) + M(BXY + YXB).
\end{aligned}$$

This proves (1).

For $X, Y \in \mathscr{A}$, write $XSY + YSX = \frac{1}{2}((2X)SY + YS(2X))$. Similar to above, it follows from the first equality of (2.2) that (2) holds, completing the proof. $\square$

By applying Lemma 1.1 (2) and [10, Lemmas 2.3–2.5], we can easily obtain the following.

Lemma 2.5. *Let $A \in \mathscr{A}$. Then*

(1) $\mathscr{F}_{11}A\mathscr{F}_{12} = 0$ *implies* $P_1AP_1 = 0$;
(2) $\mathscr{F}_{12}A\mathscr{F}_{22} = 0$ *implies* $P_2AP_2 = 0$;
(3) $\mathscr{F}_{11}A\mathscr{F}_{22} = 0$ *implies* $P_1AP_2 = 0$.

Lemma 2.6. *Let $A_{11} \in \mathscr{F}_{11}$, $A_{12} \in \mathscr{F}_{12}$ and $A_{22} \in \mathscr{F}_{22}$. Then*

(1) $M(A_{11} + A_{12} + A_{22}) = M(A_{11}) + M(A_{12}) + M(A_{22})$;
(2) $M^{*^{-1}}(A_{11} + A_{12} + A_{22}) = M^{*^{-1}}(A_{11}) + M^{*^{-1}}(A_{12}) + M^{*^{-1}}(A_{22})$.

Proof. Because of (2.2), we only need to prove (1). By the surjectivity of M, choose $S \in \mathscr{A}$ such that

$$M(S) = M(A_{11}) + M(A_{12}) + M(A_{22}).$$

Let $X_{11} \in \mathscr{F}_{11}$, $X_{12}, Y_{12} \in \mathscr{F}_{12}$ and $Y_{22} \in \mathscr{F}_{22}$ be arbitrary. Since $Y_{12}SX_{11}$=0, we see that from Lemmas 2.1 and 2.4 (2)

$$\begin{aligned}
M^{*^{-1}}(X_{11}SY_{12}) &= M^{*^{-1}}(X_{11}SY_{12} + Y_{12}SX_{11}) \\
&= M^{*^{-1}}(X_{11}A_{11}Y_{12} + Y_{12}A_{11}X_{11}) + M^{*^{-1}}(X_{11}A_{12}Y_{12} + Y_{12}A_{12}X_{11}) \\
&\quad + M^{*^{-1}}(X_{11}A_{22}Y_{12} + Y_{12}A_{22}X_{11}) \\
&= M^{*^{-1}}(X_{11}A_{11}Y_{12}).
\end{aligned}$$

Hence $X_{11}SY_{12} = X_{11}A_{11}Y_{12}$. So $P_1SP_1 = A_{11}$ by Lemma 2.5 (1).

Substituting $\{X_{11}, Y_{22}\}$, respectively $\{X_{12}, Y_{22}\}$, for $\{X_{11}, Y_{12}\}$, we can similarly get $P_1SP_2 = A_{12}$ and $P_2SP_2 = A_{22}$. Noting that $P_2SP_1 = 0$, we have

$$S = P_1SP_1 + P_1SP_2 + P_2SP_2 = A_{11} + A_{12} + A_{22},$$

as desired. □

Lemma 2.7. *Let $A_{12}, B_{12} \in \mathscr{F}_{12}$. Then*

(1) $M(A_{12} + B_{12}) = M(A_{12}) + M(B_{12})$;
(2) $M^{*^{-1}}(A_{12} + B_{12}) = M^{*^{-1}}(A_{12}) + M^{*^{-1}}(B_{12})$.

Proof. We only prove (1). To do this, suppose $S \in \mathscr{A}$ such that

$$M(S) = M(A_{12}) + M(B_{12}).$$

Let $X_{11}, Y_{11} \in \mathscr{F}_{11}$ and $X_{22} \in \mathscr{F}_{22}$ be arbitrary. Since $X_{22}SX_{11}Y_{11} = 0$, we have by Lemma 2.4 (2)

$$\begin{aligned}
M^{*^{-1}}(X_{11}Y_{11}SX_{22}) &= M^{*^{-1}}((X_{11}Y_{11})SX_{22} + X_{22}S(X_{11}Y_{11})) \\
&= M^{*^{-1}}(X_{11}Y_{11}A_{12}X_{22} + X_{22}A_{12}X_{11}Y_{11}) \\
&\quad + M^{*^{-1}}(X_{11}Y_{11}B_{12}X_{22} + X_{22}B_{12}X_{11}Y_{11}) \\
&= M^{*^{-1}}(X_{11}Y_{11}A_{12}X_{22}) + M^{*^{-1}}(X_{11}Y_{11}B_{12}X_{22}).
\end{aligned}$$

On the other hand, making use of Lemmas 2.3, 2.6 and the following equality

$$\begin{aligned}
X_{11}Y_{11}A_{12}X_{22} + X_{11}Y_{11}B_{12}X_{22} &= \frac{1}{2}(2X_{11})(Y_{11} + Y_{11}B_{12})(A_{12}X_{22} + X_{22}) \\
&\quad + \frac{1}{2}(A_{12}X_{22} + X_{22})(Y_{11} + Y_{11}B_{12})(2X_{11})
\end{aligned}$$

we can see that

$$
\begin{aligned}
&M^{*^{-1}}(X_{11}Y_{11}A_{12}X_{22} + X_{11}Y_{11}B_{12}X_{22}) \\
&\quad = \frac{1}{2}M^{*^{-1}}(2X_{11})M(Y_{11} + Y_{11}B_{12})M^{*^{-1}}(A_{12}X_{22} + X_{22}) \\
&\qquad + \frac{1}{2}M^{*^{-1}}(A_{12}X_{22} + X_{22})M(Y_{11} + Y_{11}B_{12})M^{*^{-1}}(2X_{11}) \\
&\quad = \frac{1}{2}M^{*^{-1}}(2X_{11})M(Y_{11})M^{*^{-1}}(A_{12}X_{22}) \\
&\qquad + \frac{1}{2}M^{*^{-1}}(A_{12}X_{22})M(Y_{11})M^{*^{-1}}(2X_{11}) \\
&\qquad + \frac{1}{2}M^{*^{-1}}(2X_{11})M(Y_{11})M^{*^{-1}}(X_{22}) \\
&\qquad + \frac{1}{2}M^{*^{-1}}(X_{22})M(Y_{11})M^{*^{-1}}(2X_{11}) \\
&\qquad + \frac{1}{2}M^{*^{-1}}(2X_{11})M(Y_{11}B_{12})M^{*^{-1}}(A_{12}X_{22}) \\
&\qquad + \frac{1}{2}M^{*^{-1}}(A_{12}X_{22})M(Y_{11}B_{12})M^{*^{-1}}(2X_{11}) \\
&\qquad + \frac{1}{2}M^{*^{-1}}(2X_{11})M(Y_{11}B_{12})M^{*^{-1}}(X_{22}) \\
&\qquad + \frac{1}{2}M^{*^{-1}}(X_{22})M(Y_{11}B_{12})M^{*^{-1}}(2X_{11}) \\
&\quad = M^{*^{-1}}(X_{11}Y_{11}A_{12}X_{22} + A_{12}X_{22}Y_{11}X_{11}) \\
&\qquad + M^{*^{-1}}(X_{11}Y_{11}X_{22} + X_{22}Y_{11}X_{11}) \\
&\qquad + M^{*^{-1}}(X_{11}Y_{11}B_{12}A_{12}X_{22} + A_{12}X_{22}Y_{11}B_{12}X_{11}) \\
&\qquad + M^{*^{-1}}(X_{11}Y_{11}B_{12}X_{22} + X_{22}Y_{11}B_{12}X_{11}) \\
&\quad = M^{*^{-1}}(X_{11}Y_{11}A_{12}X_{22}) + M^{*^{-1}}(X_{11}Y_{11}B_{12}X_{22}).
\end{aligned}
$$

Hence $M^{*^{-1}}(X_{11}Y_{11}SX_{22}) = M^{*^{-1}}(X_{11}Y_{11}A_{12}X_{22} + X_{11}Y_{11}B_{12}X_{22})$, and so

$$X_{11}Y_{11}SX_{22} = X_{11}Y_{11}(A_{12} + B_{12})X_{22}.$$

It follows from Lemma 2.5 (3) that $P_1SP_2 = A_{12} + B_{12}$.

Next, for any $X_{11} \in \mathscr{F}_{11}$ and $Y_{12} \in \mathscr{F}_{12}$, we have by Lemma 2.4 (2)

$$
\begin{aligned}
M^{*^{-1}}(X_{11}SY_{12}) &= M^{*^{-1}}(X_{11}SY_{12} + Y_{12}SX_{11}) \\
&= M^{*^{-1}}(X_{11}A_{12}Y_{12} + Y_{12}A_{12}X_{11}) + M^{*^{-1}}(X_{11}B_{12}Y_{12} + Y_{12}B_{12}X_{11}) \\
&= 0.
\end{aligned}
$$

Then $X_{11}SY_{12} = 0$ and moreover $P_1SP_1 = 0$ by Lemma 2.5 (1). By replacing $\{X_{11}, Y_{12}\}$ with $\{X_{12}, Y_{22}\}$ where $X_{12} \in \mathscr{F}_{12}$ and $Y_{22} \in \mathscr{F}_{22}$ are arbitrary, we similarly get $P_2SP_2 = 0$. Therefore $S = P_1SP_2 = A_{12} + B_{12}$. This completes the proof. □

Lemma 2.8. *$M(A_{11} + B_{11}) = M(A_{11}) + M(B_{11})$ for $A_{11}, B_{11} \in \mathscr{F}_{11}$.*

Proof. Choose $S \in \mathscr{A}$ such that $M(S) = M(A_{11}) + M(B_{11})$. In a similar way as in the proofs of Lemmas 2.6 and 2.7, it is easily seen that $P_1SP_2 = P_2SP_2 = 0$. So $S = P_1SP_1$.

For every $X_{11} \in \mathscr{F}_{11}$ and $Y_{12} \in \mathscr{F}_{12}$, applying Lemmas 2.4 (2) and 2.7 (2) we obtain

$$\begin{aligned}
M^{*^{-1}}(X_{11}SY_{12}) &= M^{*^{-1}}(X_{11}SY_{12} + Y_{12}SX_{11}) \\
&= M^{*^{-1}}(X_{11}A_{11}Y_{12} + Y_{12}A_{11}X_{11}) + M^{*^{-1}}(X_{11}B_{11}Y_{12} + Y_{12}B_{11}X_{11}) \\
&= M^{*^{-1}}(X_{11}A_{11}Y_{12}) + M^{*^{-1}}(X_{11}B_{11}Y_{12}) \\
&= M^{*^{-1}}(X_{11}A_{11}Y_{12} + X_{11}B_{11}Y_{12}).
\end{aligned}$$

It follows that $X_{11}SY_{12} = X_{11}(A_{11} + B_{11})Y_{12}$. Thus $P_1SP_1 = A_{11} + B_{11}$ by Lemma 2.5 (1) and then $S = A_{11} + B_{11}$. The proof is complete. □

Lemma 2.9. *$M(A_{22} + B_{22}) = M(A_{22}) + M(B_{22})$ for $A_{22}, B_{22} \in \mathscr{F}_{22}$.*

Proof. The proof is similar to that of Lemma 2.8, and it is included for the convenience of the reader. Choose $S \in \mathscr{A}$ such that $M(S) = M(A_{22}) + M(B_{22})$. Then we can easily get $P_1SP_1 = P_1SP_2 = 0$ and so $S = P_2SP_2$.

For any $X_{12} \in \mathscr{F}_{12}$ and $Y_{22} \in \mathscr{F}_{22}$, by Lemmas 2.4 (2) and 2.7 (2) we have

$$\begin{aligned}
M^{*^{-1}}(X_{12}SY_{22}) &= M^{*^{-1}}(X_{12}SY_{22} + Y_{22}SX_{12}) \\
&= M^{*^{-1}}(X_{12}A_{22}Y_{22} + Y_{22}A_{22}X_{12}) + M^{*^{-1}}(X_{12}B_{22}Y_{22} + Y_{22}B_{22}X_{12}) \\
&= M^{*^{-1}}(X_{12}A_{22}Y_{22}) + M^{*^{-1}}(X_{12}B_{22}Y_{22}) \\
&= M^{*^{-1}}(X_{12}A_{22}Y_{22} + X_{12}B_{22}Y_{22}).
\end{aligned}$$

Then $X_{12}SY_{22} = X_{12}(A_{22} + B_{22})Y_{22}$. Hence $S = P_2SP_2 = A_{22} + B_{22}$ by Lemma 2.5 (2), as desired. □

Lemma 2.10. *M is additive on $\mathscr{F}$.*

Proof. Let $A, B \in \mathscr{F}$ and write $A = A_{11} + A_{12} + A_{22}$, $B = B_{11} + B_{12} + B_{22}$. Then making use of Lemmas 2.6–2.9 we get

$$\begin{aligned}M(A+B) &= M(A_{11}+A_{12}+A_{22}+B_{11}+B_{12}+B_{22})\\ &= M(A_{11}+B_{11})+M(A_{12}+B_{12})+M(A_{22}+B_{22})\\ &= M(A_{11})+M(B_{11})+M(A_{12})+M(B_{12})+M(A_{22})+M(B_{22})\\ &= M(A)+M(B),\end{aligned}$$

completing the proof. □

It is the time to complete the proof of Theorem 2.1.

Proof of Theorem 2.1. First prove that M is additive on $\mathscr{A}$. If $\mathscr{N}=\{0,I\}$, then $\mathrm{Alg}\mathscr{N}=\mathscr{B}(H)$. In this case, $\mathscr{A}$ is clearly a 2-torsion free prime ring and contains a non-trivial idempotent rank one operator since $\dim H>1$. Thus M is additive on $\mathscr{A}$ by Proposition 1.1.

Suppose $\mathscr{N}\neq\{0,I\}$ and let $A,B\in\mathscr{A}$. Choose $S\in\mathscr{A}$ such that $M(S)=M(A)+M(B)$. Let $X,Y\in\mathscr{F}$ be arbitrary. Noting that $AXY, YXA, BXY, YXB\in\mathscr{F}$, by Lemmas 2.4 (1) and 2.10 we have

$$\begin{aligned}M(SXY+YXS) &= M(AXY+YXA)+M(BXY+YXB)\\ &= M(AXY+YXA+BXY+YXB).\end{aligned}$$

Consequently, $(S-(A+B))XY+YX(S-(A+B))=0$. Applying Lemma 1.1(2) we have $S=A+B$ immediately, as desired.

Now let us show that M^* is additive on $\mathscr{B}$. Let $U,V\in\mathscr{B}$. For all $X,Y\in\mathscr{F}$, by the additivity of M and (1.1) we have

$$\begin{aligned}&M(XM^*(U)Y+YM^*(U)X+XM^*(V)Y+YM^*(V)X)\\ &\quad= M(XM^*(U)Y+YM^*(U)X)+M(XM^*(V)Y+YM^*(V)X)\\ &\quad= \frac{1}{2}M(2X)UM(Y)+\frac{1}{2}M(Y)UM(2X)\\ &\qquad+\frac{1}{2}M(2X)VM(Y)+\frac{1}{2}M(Y)VM(2X)\\ &\quad= \frac{1}{2}M(2X)(U+V)M(Y)+\frac{1}{2}M(Y)(U+V)M(2X)\\ &\quad= M(XM^*(U+V)Y+YM^*(U+V)X).\end{aligned}$$

Then

$$\begin{aligned}&X(M^*(U)+M^*(V))Y+Y(M^*(U)+M^*(V))X\\ &\quad= XM^*(U+V)Y+YM^*(U+V)X\end{aligned}$$

and hence $M^*(U+V)=M^*(U)+M^*(V)$ by Lemma 1.1 (2) again. This completes the proof. □

Finally, we remark that with a slightly modification, Theorem 2.1 can be proved to be still true if $\frac{1}{2}$ in (1.1) is replaced by a non-zero scalar.

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